The core cover in relation to the nucleolus and the Weber set

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Abstract

The class of games for which the core coincides with the core cover (compromise stable games) is characterized. Moreover an easy explicit formula for the nucleolus for this class of games is developed, using an approach based on bankruptcy problems. Also, the class of convex and compromise stable games is characterized. The relation between the core cover and the Weber set is studied and it is proved that under a weak condition their intersection is nonempty.

Keywords: Core cover, core, Weber set, nucleolus, bankruptcy games, clan games.

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1 Introduction

An important issue in cooperative game theory is the allocation of the value of the grand coalition of a game to the players of this game. To this aim various solution concepts have been developed. They can be categorized in one point solution concepts, e.g. the Shapley value (Shapley (1953)), the nucleolus (Schmeidler (1969)) and the compromise value (Tijs (1981)), and set-valued solutions concepts, e.g. the core (Gillies (1953)), the core cover (Tijs and Lipperts (1982)) and the Weber set (Weber (1988)). The core is contained in the Weber set and the core cover. Furthermore, the nucleolus is an element of the core. It is established that a game is convex (Shapley (1971), Ichiishi (1981)) if and only if the Weber set coincides with the core.

In this paper the class of games for which the core coincides with the core cover (compromise stable games) is characterized. This class contains the class of bankruptcy games (Curiel, Maschler, and Tijs (1988)) and clan games (Potters, Poos, Muto, and Tijs (1989)). Moreover an easy explicit formula for the nucleolus for this class of games is developed, using an approach based on bankruptcy problems. As an application an easy proof of the formula for the nucleolus of clan games as derived by Potters et al. (1989) is provided. Furthermore the class of convex and compromise stable games is characterized. Finally, the relation between the core cover and the Weber set is studied. It is proved that under a weak condition their intersection is nonempty.

In section 2 we summarize some main known facts on the core cover. Section 3 deals with the characterization of the class of compromise stable games. Section 4 derives an explicit formula for the nucleolus for compromise stable games and discusses an application to clan games. The final section studies the relation between the core cover and the Weber set.

2 Core cover

This section reviews some general notions dealing with the core cover of transferable utility games. A transferable utility game (TU-game) consists of a pair (N, v), in which N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a function assigning to each coalition $S \in 2^N$ a payoff $v(S)$. By definition $v(\emptyset) = 0$. The set of all transferable utility games with player set N is denoted by TU^N.

The **core** $C(v)$ of a game $v \in TU^N$ is given by:

$$C(v) = \Big\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N),\ \sum_{i \in S} x_i \geq v(S),\ \forall S \in 2^N \backslash \{\emptyset\}\Big\}.$$

The core of a game consists of those payoff vectors such that no coalition has an incentive to split off. The core of a game might be empty.

The **utopia vector** $M(v)$ of $v \in TU^N$ consists of the utopia demands of all players. The utopia demand of player $i \in N$ is given by:

$$M_i(v) = v(N) - v(N \backslash \{i\}).$$

The **minimum right** $m_i(v)$ of player i corresponds to the minimum value this player can achieve by satisfying all other players in a coalition by giving them their utopia demands:

$$m_i(v) = \max_{S: i \in S} \Big\{v(S) - \sum_{j \in S \backslash \{i\}} M_j(v)\Big\}.$$

The **core cover** $CC(v)$ consists of all efficient payoff vectors, giving each player at least his minimum right, but no more than his utopia demand:

$$CC(v) = \Big\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = v(N),\ m(v) \leq x \leq M(v)\Big\}.$$

The elements of the core cover can be interpreted as possible allocations of the value of the grand coalition and can be seen as compromise values between $m(v)$ and $M(v)$. Note that the core cover of a game can be empty. A game $v \in TU^N$ is said to be **compromise admissible** if:

$$m(v) \leq M(v) \text{ and } \sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v).$$

Clearly the core cover of v is non-empty if and only if v is compromise admissible. The class of all compromise admissible games with player set N is denoted by CA^N. The following result about the core and the core cover is well known:

Proposition 2.1 (Tijs and Lipperts (1982)) *Let $v \in TU^N$, then $C(v) \subset CC(v)$.*

The extreme points of the core cover can be described by larginal vectors. The concept of larginals is also used in González Díaz, Borm, Hendrickx, and Quant (2003) and in Quant, Borm, Hendrickx, and Zwikker (2004). The first paper uses larginal vectors to give an alternative characterization of the compromise value. The latter paper studies the average of all larginals as a one point solution concept. An order of N is a bijective function $\sigma : \{1, \ldots, |N|\} \to N$. The player at position k in the order σ is denoted by $\sigma(k)$. The set of all orders of N is denoted by $\Pi(N)$. For $\sigma \in \Pi(N)$ the **larginal** $l^\sigma(v)$ is the efficient payoff vector giving the first players in σ their utopia demands as long as it is still possible to satisfy the remaining players with at least their minimum rights.

Definition 2.1 *Let $v \in CA^N$ and $\sigma \in \Pi(N)$. The larginal vector $l^\sigma(v)$ is defined by:*

$$l^\sigma_{\sigma(k)}(v) = \begin{cases} M_{\sigma(k)}(v) & \text{if} \quad \sum_{j=1}^{k} M_{\sigma(j)}(v) + \sum_{j=k+1}^{|N|} m_{\sigma(j)}(v) \leq v(N), \\ m_{\sigma(k)}(v) & \text{if} \quad \sum_{j=1}^{k-1} M_{\sigma(j)}(v) + \sum_{j=k}^{|N|} m_{\sigma(j)}(v) \geq v(N), \\ v(N) - \sum_{j=1}^{k-1} M_{\sigma(j)}(v) - \sum_{j=k+1}^{|N|} m_{\sigma(j)}(v) & \text{otherwise}, \end{cases}$$

for every $k \in \{1, \ldots, |N|\}$.

It is easily seen that the core cover equals the convex hull of all larginals:

$$CC(v) \quad = \quad \mathrm{conv}\big\{l^\sigma(v) \mid \sigma \in \Pi(N)\big\}.$$

The first player with respect to σ who does not receive his utopia payoff is called the **pivot** of $l^\sigma(v)$. In case every player gets his utopia payoff, we define the pivot to be the last player. Note that each larginal vector contains exactly one pivot. The following example illustrates the notion of larginal vectors and pivots.

Example 2.1 *Let $v \in CA^N$ be the game defined by:*

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v(S)$	1	0	0	0	1	2	1	2	2	2	7	4	6	8	10

Then $M(v) = (2, 4, 6, 3)$ and $m(v) = (1, 0, 1, 0)$, so $v \in CA^N$. For $\sigma =$ (1234), $l^\sigma(v)$ equals $(2, 4, 4, 0)$ and player 3 is the pivot. If $\sigma = (3421)$, the *corresponding larginal equals* $l^\sigma(v) = (1,0,6,3)$ *and player* 2 *is the pivot. The core cover of* v *is described by:*

$$\begin{aligned} CC(v) &= conv\big\{l^\sigma(v) \mid \sigma \in \Pi(N)\big\} \\ &= conv\big\{(2,4,4,0),(2,4,1,3),(2,2,6,0),(2,0,6,2),(2,0,5,3), \\ &\qquad (1,4,5,0),(1,4,2,3),(1,0,6,3),(1,3,6,0)\big\}. \end{aligned}$$

3 Core and core cover

In this section we characterize the class of compromise stable games, i.e. the class of games for which the core cover coincides with the core. Furthermore we characterize the class of convex compromise stable games.

We are interested in the class of compromise stable games. For example bankruptcy games and clan games (the precise definitions are provided later on) are compromise stable games.

Definition 3.1 *A game* $v \in CA^N$ *is* ***compromise stable*** *if* $C(v) = CC(v)$.

The following theorem characterizes the class of compromise stable games.

Theorem 3.1 *Let* $v \in CA^N$. *Then* v *is compromise stable if and only if for all* $S \in 2^N\backslash\{\emptyset\}$ *the following is true:*

$$v(S) \;\leq\; \max\Big\{\sum_{i\in S} m_i(v), v(N) - \sum_{i\in N\backslash S} M_i(v)\Big\}. \tag{1}$$

Proof: Let $v \in CA^N$. First suppose that $C(v) = CC(v)$. Then for all $\sigma \in \Pi(N)$, $l^\sigma(v) \in C(v)$. Let $S \in 2^N\backslash\{\emptyset\}$. We show that (1) is satisfied. Let $\sigma \in \Pi(N)$ begin with all players of $N\backslash S$ and end with the players of S. Hence for $k \in \{1,\ldots,|N\backslash S|\}$, $\sigma(k) \in N\backslash S$. Let $l^\sigma(v)$ be the larginal vector corresponding to σ. There are two possibilities:

- The pivot of $l^\sigma(v)$ is an element of $N\backslash S$. In this case each player of S has a payoff equal to his minimum right. We conclude that:
$$v(S) \leq \sum_{i\in S} l_i^\sigma(v) = \sum_{i\in S} m_i(v).$$
- The pivot of $l^\sigma(v)$ is an element of S. This implies that each player in $N\backslash S$ achieves a payoff equal to his utopia demand. It follows that:
$$v(S) \leq \sum_{i\in S} l_i^\sigma(v) = v(N) - \sum_{i\in N\backslash S} l_i^\sigma(v) = v(N) - \sum_{i\in N\backslash S} M_i(v).$$

Combining these two cases yields:

$$v(S) \quad \leq \quad \max\Big\{\sum_{i\in S} m_i(v), v(N) - \sum_{i\in N\setminus S} M_i(v)\Big\}.$$

Conversely, assume that inequality (1) is satisfied for each $S \in 2^N\setminus\{\emptyset\}$. By convexity of the core it suffices to show that for each order $\sigma \in \Pi(N)$, $l^\sigma(v)$ is an element of the core. Let $\sigma \in \Pi(N)$ and $S \in 2^N\setminus\{\emptyset\}$. Then:

$$\begin{aligned} v(S) &\leq \max\Big\{\sum_{i\in S} m_i(v), v(N) - \sum_{i\in N\setminus S} M_i(v)\Big\} \\ &\leq \max\Big\{\sum_{i\in S} l_i^\sigma(v), v(N) - \sum_{i\in N\setminus S} l_i^\sigma(v)\Big\} = \sum_{i\in S} l_i^\sigma(v). \end{aligned}$$

The core condition concerning coalition S is satisfied. Hence, $l^\sigma(v)$ is an element of $C(v)$. $\square$

A game $v \in TU^N$ is **convex** if for all $i \in N$ and all $S \subset T \subset N\setminus\{i\}$:

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T).$$

For convex games the marginal contribution of a player increases if this player joins a larger coalition. If $v \in TU^N$ is convex, it is easily verified that $m_i(v) = v(\{i\})$ for all $i \in N$.

In the following we focus on games which are both convex and compromise stable. A well-known class of games satisfying both convexity and compromise stability is the class of bankruptcy games (O'Neill (1982)). These games arise from so-called bankruptcy situations. **Bankruptcy situations** are formalized by a pair (E, d). Here $E \geq 0$ is the estate which has to be divided among the claimants in N and $d \in \mathbb{R}^N$, $d \geq 0$ is a vector of claims. By the nature of a bankruptcy problem $E \leq \sum_{i\in N} d_i$.

One can associate a **bankruptcy game** $v_{E,d} \in TU^N$ to a bankruptcy problem (E, d). The value of a coalition S is determined by the amount of E that is not claimed by $N\setminus S$:

$$v_{E,d}(S) = \max\Big\{0, E - \sum_{i\in N\setminus S} d_i\Big\}.$$

A game $v \in TU^N$ is **additive** if there exists a vector $a \in \mathbb{R}^N$ such that $v(S) = \sum_{i\in S} a_i$ for all $S \in 2^N$. The game v is then denoted by a. A game $v \in TU^N$ is **strategically equivalent** to $w \in TU^N$ if there exist a positive real number k and an additive game $a \in TU^N$ such that $w = a + kv$.

The next theorem states that bankruptcy games are essentially the only games that are both convex and compromise stable.

Theorem 3.2 *A game $v \in TU^N$ is both convex and compromise stable if and only if v is strategically equivalent to a bankruptcy game.*

Proof: Let $v \in TU^N$ be a convex compromise stable game. Define $a_i = v(\{i\}) = m_i(v)$ (the last equality is satisfied because v is convex) and $w(S) = v(S) - \sum_{i \in S} a_i$ for all $S \in 2^N$. Then $w \in TU^N$ is convex and compromise stable. Furthermore $m_i(w) = w(\{i\}) = 0$ for all $i \in N$ and $C(w) = CC(w)$. Furthermore:

$$M(w) = M(v) - m(v) \text{ and } m(w) = 0.$$

We show that w is the bankruptcy game $v_{E,d}$ with $E = w(N)$ and $d = M(w)$. For $S \in 2^N \backslash \{\emptyset\}$:

$$v_{E,d}(S) = \max\Big\{0, E - \sum_{i \in N \backslash S} M_i(w)\Big\} = \max\Big\{\sum_{i \in S} m_i(w), E - \sum_{i \in N \backslash S} M_i(w)\Big\}.$$

Theorem 3.1 implies $w(S) \leq v_{E,d}(S)$ for all $S \subset N$. Now suppose there is a coalition $S \in 2^N \backslash \{\emptyset\}$ such that $w(S) < v_{E,d}(S)$. Because w is convex, $w(S) \geq \sum_{i \in S} w(\{i\}) = \sum_{i \in S} m_i(w)$ and hence:

$$w(S) < E - \sum_{i \in N \backslash S} M_i(w) = w(N) - \sum_{i \in N \backslash S} M_i(w).$$

Consider $\sigma \in \Pi(N)$ that begins with the players of S and ends with the players of $N \backslash S$, i.e. $\sigma(k) \in S$ for $k \in \{1, \ldots, |S|\}$. The payoff of coalition $N \backslash S$ according to the marginal vector $m^\sigma(w)$ is given by:

$$\sum_{j \in N \backslash S} m_j^\sigma(w) = w(N) - w(S) > \sum_{j \in N \backslash S} M_j(w).$$

This implies that $m^\sigma(w) \notin CC(w)$. This contradicts $CC(w) = C(w)$.

The converse is also true because bankruptcy games are convex games and the core of a bankruptcy game coincides with the core cover (cf. Curiel et al. (1988)). $\square$

It is trivial to show that for any 3-player TU-game the core cover equals the core. From Theorem 3.2 it then follows that each convex three player game is strategically equivalent to a bankruptcy game.

4 The nucleolus of compromise stable games

This section analyzes the nucleolus of compromise stable games: it develops a formula which is based on the Talmud rule for bankruptcy problems.

Let (E, d) be a bankruptcy problem. The **constrained equal award rule** (CEA) is for all $i \in N$ defined by

$$CEA_i(E, d) = \min\{\alpha, d_i\},$$

with α such that $\sum_{i \in N} \min\{\alpha, d_i\} = E$. The **Talmud rule** (TAL) (cf. Aumann and Maschler (1985)) is defined as

$$TAL_i(E, d) \quad = \quad \begin{cases} CEA_i(E, \frac{1}{2}d) & \text{if } \sum\limits_{j \in N} d_j \geq 2E \\ d_i - CEA_i\big(\sum_{j \in N} d_j - E, \frac{1}{2}d\big) & \text{if } \sum\limits_{j \in N} d_j < 2E \end{cases}$$

for all $i \in N$. Aumann and Maschler (1985) prove that the Talmud rule equals the nucleolus (cf. Schmeidler (1969)) of the corresponding bankruptcy game. The **nucleolus** of a game[1] $v \in TU^N$ is denoted by $\nu(v)$. For our results we do not need the exact definition of the nucleolus, but we only use the following important result.

Theorem 4.1 (Potters and Tijs (1994)) *Let $v, w \in TU^N$ be such that v is convex and $C(v) = C(w)$. Then $\nu(v) = \nu(w)$.*

The following theorem shows that the nucleolus for compromise stable games can be computed by first giving every player his minimum right and then adding the value of the Talmud rule of a bankruptcy problem derived from the corresponding game.

Theorem 4.2 *Let $v \in CA^N$ be compromise stable. Then*

$$\nu(v) \quad = \quad m(v) + TAL\Big(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\Big). \tag{2}$$

Proof: Let $v \in CA^N$ be compromise stable. Define the additive game $a \in TU^N$ by taking $a_i = m_i(v)$ for all $i \in N$, and define $w \in TU^N$ as $w(S) = v(S) - \sum_{i \in S} a_i$, $S \in 2^N$. Because the nucleolus is relative invariant with respect to strategic equivalence we have

$$\nu(v) = a + \nu(w) = m(v) + \nu(w).$$

[1]In fact, the game should have a non-empty imputation set.

For w the following assertions can easily be verified $M(w) = M(v) - m(v)$, $m(w) = 0$, $w(N) = v(N) - \sum_{i \in N} m_i(v)$, and $C(w) = CC(w)$.

Consider the bankruptcy problem defined by $E = w(N)$ and $d = M(w)$. For the corresponding bankruptcy game $v_{E,d}$ it is true that $v_{E,d}(N) = w(N)$. By definition of $v_{E,d}$, $M_i(v_{E,d}) = \min\{E, d_i\}$, and using the convexity of $v_{E,d}$,

$$\begin{aligned} m_i(v_{E,d}) &= v_{E,d}(\{i\}) = \Big(E - \sum_{j \in N \setminus \{i\}} d_j\Big)_+ \\ &= \Big(w(N) - \sum_{j \in N \setminus \{i\}} M_j(w)\Big)_+ = 0. \end{aligned}$$

The last equality follows from the fact that $m_i(w) = 0$, and $m_i(w) \geq w(N) - \sum_{j \in N \setminus \{i\}} M_j(w)$. The core of $v_{E,d}$ can now be written as

$$\begin{aligned} C(v_{E,d}) &= CC(v_{E,d}) \\ &= \Big\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = E,\ 0 \leq x_i \leq \min\{E, d_i\},\ \forall i \in N\Big\} \\ &= \Big\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = w(N), \\ &\qquad\qquad 0 \leq x_i \leq \min\big\{w(N), M_i(w)\big\},\ \forall i \in N\Big\} \\ &= \Big\{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i = w(N),\ 0 \leq x \leq M(w)\Big\} \\ &= CC(w) = C(w). \end{aligned}$$

Since $v_{E,d}$ and w have the same core, and $v_{E,d}$ is convex, we can apply Theorem 4.1. Hence,

$$\begin{aligned} \nu(w) &= \nu(v_{E,d}) = TAL\big(E, d\big) = TAL\big(w(N), M(w)\big) \\ &= TAL\big(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\big). \end{aligned}$$

Consequently,

$$\begin{aligned} \nu(v) &= m(v) + \nu(w) \\ &= m(v) + TAL\Big(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\Big). \end{aligned}$$ $\square$

Corollary 4.1 *Let v be a 3-player game with a non-empty core. Then*

$$\nu(v) = m(v) + TAL\Big(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\Big).$$

Example 4.1 *Consider the game of Example 2.1. Then* $M(v) = (2, 4, 6, 3)$ *and* $m(v) = (1, 0, 1, 0)$. *For every coalition* S *inequality (1) is valid. For example* $v(\{1, 2\}) \leq m_1(v) + m_2(v)$ *and* $v(\{2, 3\}) \leq v(N) - M_1(v) - M_4(v)$. *Applying Theorem 3.1 we find* $C(v) = CC(v)$. *Using Theorem 4.2, the nucleolus of* v *is given by:*

$$\begin{aligned}
\nu(v) &= m(v) + TAL\Big(v(N) - \sum_{i \in N} m_i(v), M(v) - m(v)\Big) \\
&= (1, 0, 1, 0) + TAL\big(8, (1, 4, 5, 3)\big) \\
&= (1, 0, 1, 0) + (1, 4, 5, 3) - CEA\big(5, (\tfrac{1}{2}, 2, 2\tfrac{1}{2}, 1\tfrac{1}{2})\big) \\
&= (2, 4, 6, 3) - (\tfrac{1}{2}, 1\tfrac{1}{2}, 1\tfrac{1}{2}, 1\tfrac{1}{2}) = (1\tfrac{1}{2}, 2\tfrac{1}{2}, 4\tfrac{1}{2}, 1\tfrac{1}{2}).
\end{aligned}$$

We now consider the application of Theorem 3.1 and Theorem 4.2 with respect to clan games. In a **clan game** a coalition can not make any profit if a certain group (CLAN) is not part of this coalition. A game $v \in TU^N$ is a clan game if $v(S) \geq 0$ for all $S \in 2^N$, $M_i(v) \geq 0$ for all $i \in N$ and if there exists a nonempty coalition CLAN $\subset N$ such that:

(i) $v(S) = 0$ if CLAN $\not\subset S$

(ii) $v(N) - v(S) \geq \sum_{i \in N \setminus S} M_i(v)$, for all S with CLAN $\subset S$.

The last property is also known as the union property. Clan games for which CLAN $= \{i^*\}$ are also known as big boss games.[2] In the following corollary several (known) properties of clan games are easily proved with the aid of Theorems 3.1 and 4.2.

Corollary 4.2 (cf. Potters et al. (1989)) *Let* $v \in TU^N$ *be a clan game with* $|CLAN| \geq 2$. *Then* $v \in CA^N$, $C(v) = CC(v)$ *and*

$$\nu(v) = CEA\Big(v(N), \tfrac{1}{2}M(v)\Big).$$

Proof: Let $v \in TU^N$ be a clan game, with $|\text{CLAN}| \geq 2$. Then $M_i(v) = v(N)$ if $i \in$ CLAN. Let $i \in N$ and $S \subset N$ such that $i \in S$. If CLAN $\subset S$ it can be deduced from the union property that:

$$v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \leq v(N) - \sum_{j \in N \setminus \{i\}} M_j(v) \leq 0.$$

[2] This definition differs from the definition of big boss games given in Muto et al. (1988) in the sense that it is now required that $v(S) \geq 0$ for all $S \in 2^N$ and the requirement of monotonicity is weakened to $M(v) \geq 0$. A game $v \in TU^N$ is monotonic, if $v(S) \leq v(T)$ if $S \subset T$.

The last inequality follows from $M(v) \geq 0$. Since $v(S) = 0$ if CLAN $\not\subset S$, it follows that (by taking $S = \{i\}$) $m_i(v) = 0$ for all $i \in N$. Therefore $m(v) \leq M(v)$. Because $v(N) \geq 0$ and $M(v) \geq 0$ it is true that $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$. Hence $v \in CA^N$.

Let $S \in 2^N \backslash \{\emptyset\}$. If CLAN $\subset S$, then (1) is satisfied by condition (ii). If CLAN $\not\subset S$, then $v(S) = 0$ and formula (1) follows from $m(v) = 0$. Theorem 3.1 yields $C(v) = CC(v)$. Since $|CLAN| \geq 2$, we have that $\sum_{i \in N} M_i(v) \geq 2v(N)$. Hence by Theorem 4.2 and the definition of Talmud rule,

$$\nu(v) \quad = \quad CEA\Big(v(N), \tfrac{1}{2}M(v)\Big).$$

□

Note that the results of Theorem 3.1 and Theorem 4.2 can also be used to reprove the following corollary in a relatively straightforward way.

Corollary 4.3 (cf. Muto et al. (1988)) *Let $v \in TU^N$ be a clan game with $CLAN = \{i^*\}$. Then $v \in CA^N$, $C(v) = CC(v)$, and*

$$\nu_j(v) \quad = \quad \begin{cases} \frac{1}{2}M_j(v) & \text{if } j \in N\backslash\{i^*\} \\ v(N) - \frac{1}{2}\sum_{k \in N\backslash\{i^*\}} M_k(v) & \text{if } j = i^*. \end{cases}$$

The following Venn diagram summarizes the relations between the different classes of games we have encountered.

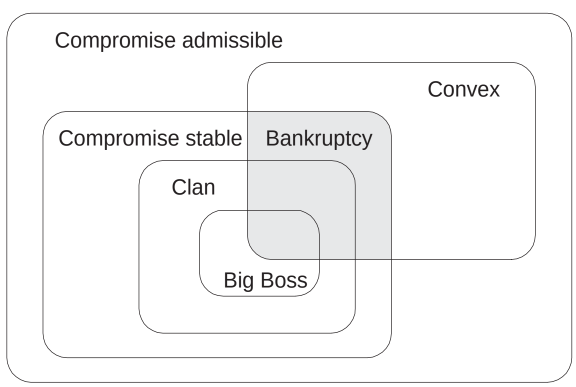

Figure 1: A Venn diagram depicting the relations between several classes of games (up to strategic equivalence).

5 Core cover and Weber set

This section studies the relation between the core cover and the Weber set.

For $\sigma \in \Pi(N)$ the corresponding **marginal vector** $m^\sigma(v)$ measures the marginal contribution of the players with respect to σ, i.e.

$$m^\sigma_{\sigma(i)}(v) = v\big(\{\sigma(1), \ldots, \sigma(i)\}\big) - v\big(\{\sigma(1), \ldots, \sigma(i-1)\}\big), \quad i \in \{1, \ldots, n\}.$$

The **Weber set** is the convex hull of all marginal vectors:

$$W(v) = \text{conv}\{m^\sigma(v) \mid \sigma \in \Pi(N)\}.$$

An important relation between core and Weber set is given in the following proposition:

Proposition 5.1 (Weber (1988)) *Let $v \in TU^N$. Then $C(v) \subset W(v)$.*

Moreover,

Proposition 5.2 (Shapley (1971) and Ichiishi (1981)) *Let $v \in TU^N$. Then v is convex if and only if $C(v) = W(v)$.*

For any TU-game the intersection of the core cover and the Weber set always contains the core. Hence, the core cover and the Weber set have points in common if the core is non-empty. This raises the question whether the intersection of the core cover and the Weber set is non-empty in general for compromise admissible games. It is showed that under a weak condition the answer is affirmative. For the proof of this theorem the following lemma is needed:

Lemma 5.1 *If $n \in \mathbb{N}$ and $d, y \in \mathbb{R}^n$ with:*

$$y_1 \geq \ldots \geq y_n, \tag{3}$$

$$\sum_{i=1}^{k} d_i \leq 0 \quad \textit{for all } k \in \{1, \ldots, n-1\}, \tag{4}$$

$$\textit{and} \quad \sum_{i=1}^{n} d_i = 0, \tag{5}$$

then,

$$d \cdot y = \sum_{i=1}^{n} d_i y_i \leq 0.$$

Proof: The proof is given by an induction argument to n. For $n = 1$ the assertion is true, since $d_1 = 0$. Assume that the lemma is satisfied for $k = n - 1$. Let $y, d \in \mathbb{R}^n$ be such that the formulas (3)–(5) are true. One can conclude that:

$$\begin{aligned} \sum_{i=1}^{n} d_i y_i &= \sum_{i=1}^{n-2} d_i y_i + d_{n-1} y_{n-1} + d_n y_{n-1} + d_n (y_n - y_{n-1}) \\ &= (\sum_{i=1}^{n-2} d_i y_i + (d_{n-1} + d_n) y_{n-1}) + d_n (y_n - y_{n-1}) \\ &\leq 0 + d_n (y_n - y_{n-1}) \leq 0. \end{aligned}$$

The first inequality follows from the induction hypothesis and the second inequality follows from the fact that $d_n \geq 0$ and $y_n - y_{n-1} \leq 0$. □

Theorem 5.1 *Let $v \in CA^N$ be such that for all $S \in 2^N$,*

$$v(S) + \sum_{j \in N \setminus S} m_j(v) \quad \leq \quad v(N). \tag{6}$$

Then $CC(v) \cap W(v) \neq \emptyset$.

Proof: Let $v \in CA^N$ be such that for all $S \in 2^N$ (6) is satisfied. Suppose that $CC(v) \cap W(v) = \emptyset$. Since $CC(v)$ and $W(v)$ are both closed and convex sets we can separate these sets with a hyperplane. This means that there exists a vector $y \in \mathbb{R}^N$ such that:

$$m \cdot y > l \cdot y \quad \text{for all } m \in W(v), \;\; l \in CC(v). \tag{7}$$

Let $\sigma \in \Pi(N)$ an order such that $y_{\sigma(1)} \geq y_{\sigma(2)} \geq \ldots \geq y_{\sigma(n)}$. Consider $l^\sigma(v)$ and $m^\sigma(v)$. Then:

$$\begin{aligned} m^\sigma(v) \cdot y - l^\sigma(v) \cdot y &= (m^\sigma(v) - l^\sigma(v)) \cdot y \\ &= \sum_{k=1}^{|N|} \big(m^\sigma_{\sigma(k)}(v) - l^\sigma_{\sigma(k)}(v)\big) y_{\sigma(k)}. \end{aligned}$$

Now we first derive some inequalities with respect to $v(S)$. Because v is compromise admissible and hence $m(v) \leq M(v)$, it is true that for all $i \in N$ and for all $S \subset N$ with $i \in S$:

$$v(S) - \sum_{j \in S \setminus \{i\}} M_j(v) \leq \max_{T : i \in T} \big\{ v(T) - \sum_{j \in T \setminus \{i\}} M_j(v) \big\} = m_i(v) \leq M_i(v).$$

This yields that for all $S \in 2^N$,

$$v(S) \leq \sum_{i \in S} M_i(v). \tag{8}$$

From (6) it follows that:

$$v(S) \leq v(N) - \sum_{j \in N \setminus S} m_j(v). \tag{9}$$

Define $d_{\sigma(k)} = m^{\sigma}_{\sigma(k)}(v) - l^{\sigma}_{\sigma(k)}(v)$ for all $k \in \{1, \ldots, |N|\}$. Then for all $r \in \{1, \ldots, |N|\}$

$$\begin{aligned} \sum_{k=1}^{r} d_{\sigma(k)} &= \sum_{k=1}^{r} \left(m^{\sigma}_{\sigma(k)}(v) - l^{\sigma}_{\sigma(k)}(v)\right) \\ &= v\left(\{\sigma(1), \cdots, \sigma(r)\}\right) - \sum_{k=1}^{r} l^{\sigma}_{\sigma(k)}(v) \leq 0. \end{aligned}$$

The inequality follows from inequalities (8) and (9), since $\sum_{k=1}^{r} l^{\sigma}_{\sigma(k)} = \sum_{k=1}^{r} M_{\sigma(k)}(v)$ or $\sum_{k=1}^{r} l^{\sigma}_{\sigma(k)} = v(N) - \sum_{k=r+1}^{|N|} m_{\sigma(k)}(v)$. Furthermore $\sum_{k=1}^{|N|} d_{\sigma(k)} = v(N) - v(N) = 0$. Applying Lemma 5.1 gives:

$$\sum_{k=1}^{|N|} d_{\sigma(k)} y_{\sigma(k)} = \sum_{k=1}^{|N|} \left(m^{\sigma}_{\sigma(k)}(v) - l^{\sigma}_{\sigma(k)}(v)\right) y_{\sigma(k)} \leq 0.$$

Hence $m^{\sigma}(v) \cdot y \leq l^{\sigma}(v) \cdot y$. This contradicts (7). □

The following example shows that it is possible that the core cover and the Weber set do not have any points in common.

Example 5.1 *Let $v \in TU^N$ and $N = \{1, \ldots, 5\}$. Let v be such that the players 1, 2 and 3 are symmetric and so are players 4 and 5. To simplify notations we say that the players 1, 2 and 3 are of type a and 4 and 5 of type b. For example the coalition $\{abb\}$ represents the coalitions $\{145\}$, $\{245\}$ or $\{345\}$. The game v is given by:*

S	a	b	aa	ab	bb	aaa	aab	abb	$aaab$	$aabb$	N
$v(S)$	0	0	-1	0	2	-1	-1	2	-1	1	1

It is easily verified that $M(v) = (0,0,0,2,2)$ and $m(v) = (0,0,0,0,0)$. Hence the core cover of v is given by:

$$CC(v) = \left\{x \in \mathbb{R}^N \mid x \geq 0,\ x_4 + x_5 = 1,\ x_1 = x_2 = x_3 = 0\right\}.$$

Because of symmetry, one does not need to calculate all marginal vectors to compute the Weber set. There are only six marginal vectors each corresponding to twenty different orders. The Weber set is given by:

$$\begin{aligned} W(v) &= conv\big\{(-1,0,0,2,0),(-1,0,0,0,2),(0,-1,0,2,0),\\ &\qquad (0,-1,0,0,2),(0,0,-1,2,0),(0,0,-1,0,2)\big\}. \end{aligned}$$

We conclude that $m_1^\sigma + m_2^\sigma + m_3^\sigma = -1$ for all $\sigma \in \Pi(N)$. Hence $m_1 + m_2 + m_3 = -1$ for all $m \in W(v)$, and therefore $CC(v) \cap W(v) = \emptyset$.

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